12-chord phrases, either of which can repeat without disturbing anything (Fig. 6 and 7).

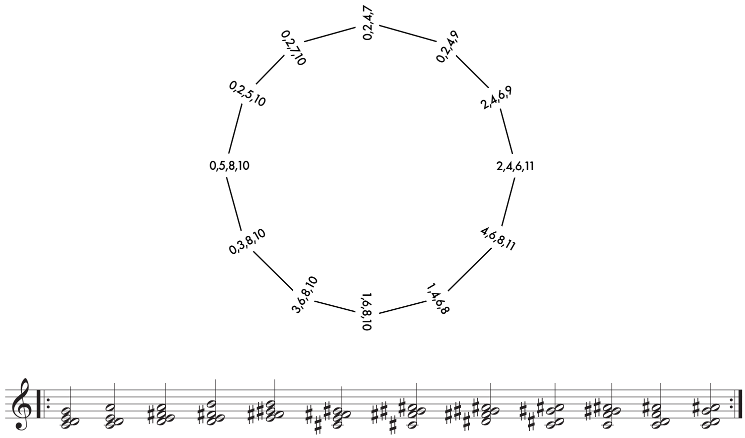


Fig. 6. Loop of 12 chords belonging to Forte 4-22

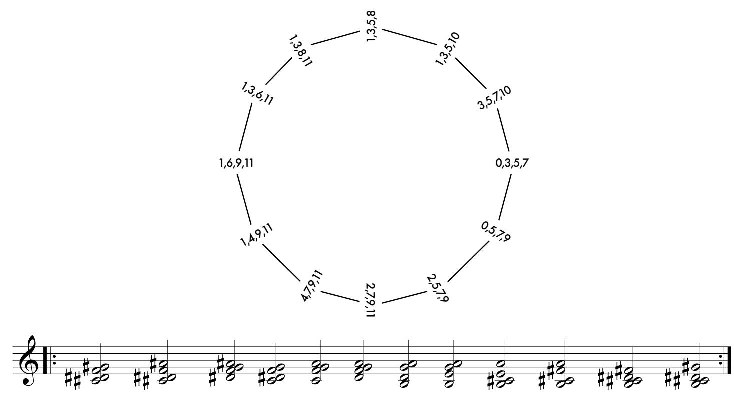


Fig. 7. The other loop with the other 12 chords from Forte 4-22

Note that the chords always alternate between the original (0,2,4,7) form and the inverted (0,3,5,7) form, and you can hear a sort of back and forth as the music goes on. The inveruse all 12 notes before repeating something, and his technique was in some ways much freer than that of the Second Viennese School. But he was rigorous in his own way, and sometimes his music is quite subtle.

He didn't always work with sets of six notes. A nice example of four-note sets can be found in the *Zwölftonspiel* (*February 1, 1946)* (Fig. 1).

Fig. 1. First twelve bars of *Zwölftonspiel*
(February 1, 1946), mov. 1

Each four-note set is presented in one bar, or two at the end of each system, and shares three notes from the previous set, adding one new note. The process continues for 14 measures, at the end of which the succession of 12 new notes has covered the whole chromatic. It is hard to see this organization in the score itself, so I made the drawing of Fig. 2.

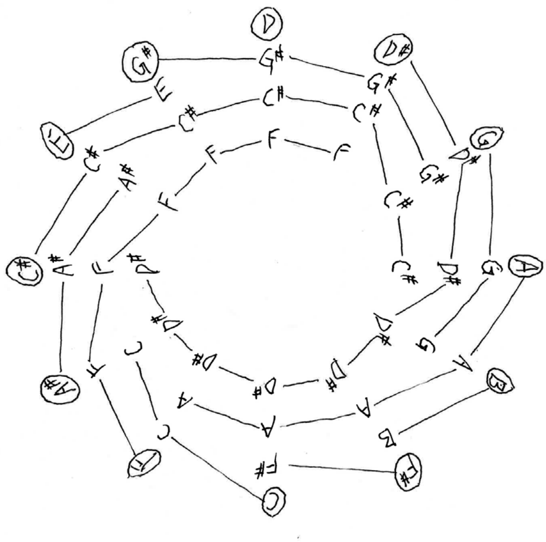


Fig. 2. First twelve bars of *Zwölftonspiel (February 1, 1946), mov. 1,* beginning with the [C♯, D♯, G, A] chord at three o'clock.

The 12-tone row of new notes forms the outer circle, the other three notes of each chord are shown on the inner circles, and the lines show how notes remain from one measure to the other. If all the notes were used the same number of times, as in an ideal atonal music, we would find each note four times, but here we find F seven times and E only twice. The important thing though, which is without exception, is that with each new measure the composer retains three notes, drops one, and adds another, and this is something one would never find in serial music. I was struck to find things like this in Hauer, because in my own work I have also been very interested in counting the number of notes that are the same or different from one chord to another, and I didn't know that someone else was doing the same thing 50 years earlier.

CONTINUATION

* Form a scale by alternating minor thirds and minor seconds. How long must you continue in order to have all 12 pitch classes of the chromatic scale?

* Form another scale where the notes progress minor third, minor third, major second, minor third, minor third, major second, etc. How many octaves must you go before returning to the original pitch class?

* Form a Slonimskian scale of two or more octaves following the preferences of Euler, with the close intervals high and the large intervals low. Fix the second note a sixth higher than the first note and make each subsequent interval narrower than the previous one.

** Form a Slonimskian scale of two octaves where there is only one difference between the notes of the two octaves. Do you think there already exists somewhere a piece of music that follows such a scale?

*** How many scales of nine notes are possible using only major seconds and minor thirds with no octaves? Use my Mathematica® program if you have the software to do it:

```
ct = 0;
Do [ scale = { 0,
i,
i + j,
i + j + k,
i + j + k + l,
i + j + k + l + m,
i + j + k + l + m + n,
i + j + k + l + m + n + o,
i + j + k + l + m + n + o + p};
```

I first became aware of Schillinger many years ago in a library in Colorado when I ran across the *The Schillinger System of Musical Composition*, a much longer book, published posthumously by Carl Fischer in 1946, and still in print. It is the longest and best known Schillinger book, and I particularly remember learning about isorhythm from reading it, a lesson I never forgot and one that was most useful for me in later years. My university professors later on knew about isorhythm, but they mostly considered it a historical phenomenon found occasionally in medieval manuscripts. For Schillinger, however, isorhythmic structures could be constructed in many new and different ways and could be most useful for composers today.

Schillinger's explanation of isorhythm is concise and simple and consists mostly of the following example, which I quote directly from the book. The melodic cycle, technically the *color*, is shown here in Fig. 1, followed by the rhythmic cycle in Fig. 2, technically the *talea*, followed by the six bars of music that result by letting the seven notes of the rhythm turn around in combination with the six notes of the melodic cycle (Fig. 3).

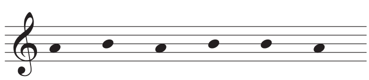


Fig. 1. Color

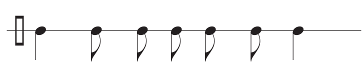


Fig. 2. Talea

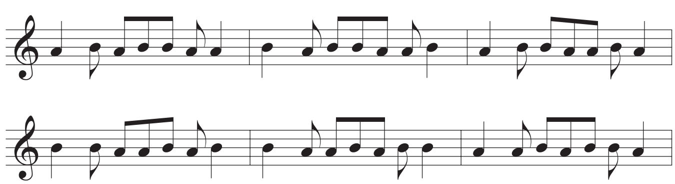


Fig. 3. Color and talea rotating together

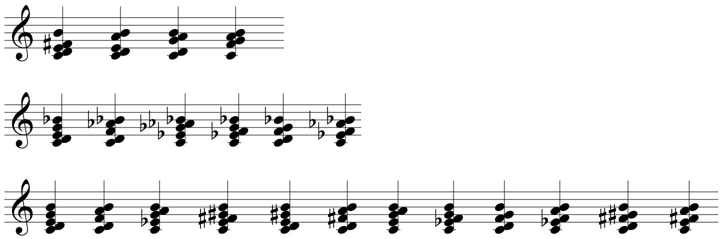


Fig. 4. Families of chords ("pentads") given by Schillinger at the end of his *Kaleidophone*

Those first four chords are the four ways that one can construct a five-note chord if the intervals between adjacent notes consist of three major seconds and one fourth. The second family, the subsequent six chords, are constructed with two major seconds and two minor thirds as adjacent intervals, and the third family has adjacent intervals of two major seconds, one minor third and one major third. The families get larger and larger because, as you know if you have studied the theory of combinations a bit, there are four ways of ordering (a,a,a,b), six ways of ordering (a,a,b,b), and 12 ways of ordering (a,a,b,c)[2]. Of course, in all three families, the top and bottom voices never move, because the sum of the intervals is always the same.

Finding that Schillinger was working with combinations of adjacent intervals was another surprise for me, because, without ever having seen this book, and working half a century after it was published, I also often calculated groups of chords according to the intervals found between adjacent notes.

Classifying families in this way is rather different from Forte's classifications. The first and seventh chords in Schil-

2. See Math Addendum.

linger's third list, where adjacent intervals are 2, 2, 3, and 4, are members of the set Forte 5-27 (0,1,3,5,8), but to make a comparison I wrote out the 10 chords of the Forte 5-27 set that have C as the bass note (Fig. 5).

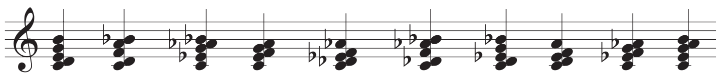


Fig. 5. From the Forte 5-27 set

Which of these two families of five-note chords form the most tightly knit family? I have the feeling that Schillinger's family is the tightest, but perhaps mostly just because they all have B as the top note. All the chords in the Forte set, on the other hand, have a kind of modern jazz quality, which is not true of the Schillinger chords. Both sets of chords are rather close families in any case.

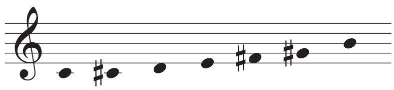
The question of chord families takes me back to my studies with Morton Feldman[3], who sometimes also spoke of "families", listening to chords that seemed to belong together and asking why some chord that appeared to be very similar just didn't sound related to the others. If it would be possible to show these two "families" to Feldman, he would probably play them over about 20 times, listen very carefully, then say something like, "Well, I'd eliminate the third and fifth chord and add this one", and play some other chord that didn't have the same intervals at all, but which somehow had a very similar sound. In that way he would make his own Feldman family.

3. Two programs are devoted to Morton Feldman in the Music by my Friends series [mbmf.editions75.com].

CONTINUATION

* Form five-note chords using this scale from Table I of Schillinger's *Kaleidophone:*

* Calculate all the three-note chords that can be derived from the following seven-note scale that Schillinger defines in this way on page 49 of his *Kaleidophone:*

1 + 1 + 2 + 2 + 2 + 3 

** Consider what might be the best sequence for these chords.

** How many four-note chords can be constructed on this scale with C as the lowest note and G♯ as the highest?

** Construct a chord family where the intervals between adjacent notes are 1, 1, and 4. How many members are in this family?

*** Define all the five-note chords possible with adjacent intervals of 2, 3, 3, and 4 semitones. Use my Mathematica® program if you have the software. To change the four intervals between the five notes, give "int" (for "intervals") other numbers to permute.

```
int = Permutations[{2, 3, 3, 4}];
Do[scale = {0, Part[int[[i]], 1],
Part[int[[i]], 1] +
Part[int[[i]], 2],
Part[int[[i]], 1] +
Part[int[[i]], 2] +
Part[int[[i]], 3],
Part[int[[i]], 1] +
Part[int[[i]], 2] +
```

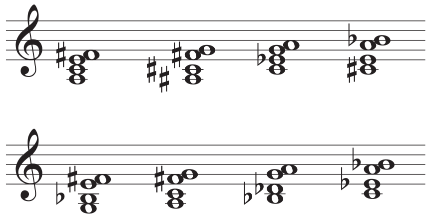


Fig. 3. More chords derived from Messiaen's Mode 2

Since Mode 3 uses a nine-note scale with three transpositions, one runs through three rather different types of chords before coming back to a transposition of the first one (Fig. 4).

Fig. 4. Chords derived from Messiaen's Mode 3
[Messiaen 1944, ex. 329]

Much the same can be said of Messiaen's four-voice example in Mode 4. Since Mode 4 has only eight notes, one forms a sequence of four chords before repeating. With the voices beginning as shown here, they all rise a half step from the first chord to the second, so there is no change. The third and fourth chords are rather different, but still, the sequence is homogeneous because everything comes from the same scale.

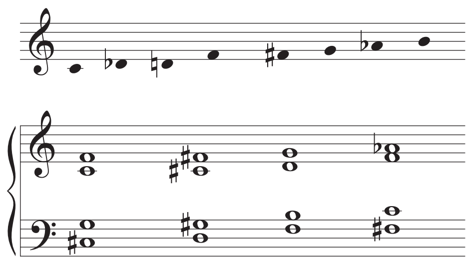


Fig. 5. Chords derived from Messiaen's Mode 4
[Messiaen 1944, ex. 345]

For Mode 6 we can write an example in eight voices, so the entire scale is present in each chord. That means that each chord is essentially the same, and the sequence is not moving in cycles as in the other cases. Maybe it is not moving at all.

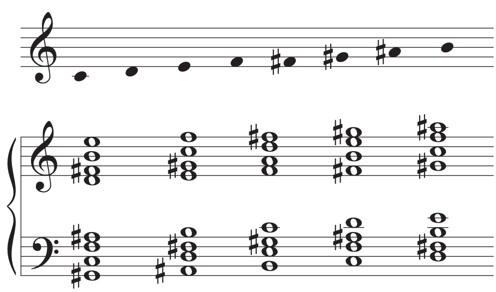


Fig. 6. Chords derived from Messiaen's Mode 6

Messiaen counts seven modes of limited transposition, but Guerino Mazzola, who counts possibilities in a more rigorous mathematical way, finds a couple of others[2] [Mazzola, 2002]. The one below, for example, has four transpositions and is really about the same as mode 3, except that it has only

2. Mazzola counts 10 scales of limited transposition instead of 7: "Es gibt 10 Skalen, von denen Messiaen aus unerfindlichen Gründen die Skalen (c, cis, dis, fis, g a) sowie deren Umkehr und (c, cis e, f, gis, a) nicht aufzählt." [*Geometrie der Töne*, p. 98].

six notes instead of nine. It is clearly a mode of limited transposition though, and Messiaen could have added it to his list. In any case the resulting sequence of four-note chords certainly sounds like Messiaen even though it consists uniquely of major seventh chords, Forte 4-20 (0,1,5,8).

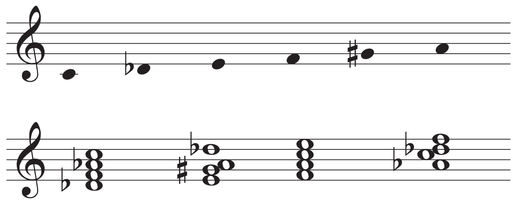


Fig. 7. Chords derived from a Mazzola mode

One might also take the eight notes of Mode 4, substitute E for D and F, substitute B♭ for A♭ and B, and have a new six-note mode of limited transposition, as I did here. There are now six transpositions possible, but the result still sounds like Messiaen, it seems to me.

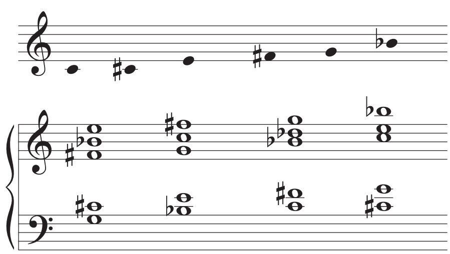


Fig. 8. More modes with limited transpositions

Volume VII of the *Traité* also discusses “renversements”, “accords tournants”, and other harmonic techniques, but the important thing here is to observe how Messiaen’s search for Other harmony is similar to that of other theorists of his time. Messiaen’s manner of deriving chords from scales is not so different from Hauer’s manner of deriving chords from family of chords that have Forte 3-7 equality, and it is complete as well. If I stopped before the 24th chord, or if I repeated some chords and not others, the logic wouldn't have been so nice, and I really believe that we can perceive completeness when we hear a sequence like this, at least on some subconscious level.

But so far we are only talking about one particular pitch class set, Forte 3-7. How can we form other families of harmonies, and how can we be sure when the family is complete? Usually deciding whether a musical sequence is complete is a subjective evaluation, but when we have finite lists of things, this decision can be perfectly objective. Of course, the problem is easier with short lists. If we want all the possible chords of three notes that can be formed with a scale of six notes, everything will be over after $6!/(3!\cdot 3!) = 20$ chords. But we can easily make longer lists. Or we can put together all the four-note chords possible with Messiaen's mode 2, or all the three-note chords that can be extracted from one of Hauer's six-note tropes, or all the five-note chords with C as the lowest note and the G an octave and a half higher as the highest note. Or all the three-note chords one can construct on a whole tone scale two octaves long. How many are there in all these cases? If you don't know how to deduce the answers to little problems like this, you can just count the possibilities in a reasonable amount of time, or try to find a mathematical way of doing it more quickly[6]. But this is enough generalities about the essentials of families, equality and completeness. It's better to go on to specific types of families one by one in subsequent chapters.

---

6. The Math Addendum at the back of this book gives tools to solve some of these problems.

As we might have expected, the possibilities are all evenly distributed, so we get a symmetrical bell-like curve. All 220 three-note chords are there, all 15 of Forte's three-note categories, all mixed up. The largest categories are the chords with sums of 15, 16, 17, and 18, with 15 chords in each category, and the graph is perfectly symmetrical, ending with only two chords having sums of 28, one with a sum of 29, and one [9, 10, 11] with a sum of 30. Let's just look at the 15 chords with sums of 16 and see what they have in common and how they might fit together musically.

```
{{4, 5, 7}, {3, 6, 7}, {3, 5, 8},
 {2, 6, 8}, {1, 7, 8}, {3, 4, 9},
 {2, 5, 9}, {1, 6, 9}, {0, 7, 9},
 {2, 4, 10}, {1, 5, 10}, {0, 6, 10},
 {2, 3, 11}, {1, 4, 11}, {0, 5, 11}}
```

We can probably understand this best if we make a graph connecting the chords by minimal differences. Since they all have the same sum, that means that to get from one chord to another, one voice has to ascend a notch and another has to descend a notch. To figure this out and find the smoothest progression, it's best to make a graph, but in this case I could figure it out without going to the trouble, and here is the progression I came up with (Fig. 2). Can you hear that it is all hovering at one altitude?

Fig. 2. Chords with sums of 16 joined by minimal differences

You may or may not like the fact that we have a lot of minor triads here. You'll understand better why when we talk about

In the following example the top voice rocks back and forth between its two notes at every change, while a second voice rocks back and forth every two changes, a third voice rocks every four changes, and a fourth voice moves very slowly, rocking only once every eight changes. The sequence here begins with all voices in their top positions and ends with all voices in their bottom positions, and it's a complete family of 16.

Fig. 9. Four voices changing in whole steps

Here is another simple example where voices rise and fall by major seconds, and no other explanation is necessary. Of course, in this case the music is getting slightly higher, little by little, then advancing downward toward a goal, which it reaches when it returns to the original chord, and this leads us to the next chapter.

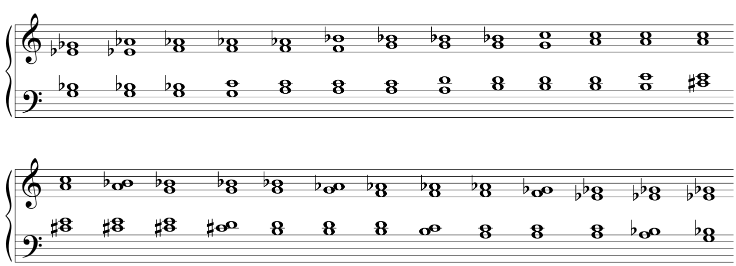


Fig. 10. Rising and falling sequence with four voices

```
Do[Print[{a, b, c}],
      {c, 3, 6},
      {b, 2, c - 1},
      {a, 1, b - 1}]

Formatted output=
   [1, 2, 3]
   [1, 2, 4]
   [1, 3, 4]
   [2, 3, 4]
   [1, 2, 5]
   [1, 3, 5]
   [2, 3, 5]
   [1, 4, 5]
   [2, 4, 5]
   [3, 4, 5]
   [1, 2, 6]
   [1, 3, 6]
   [2, 3, 6]
   [1, 4, 6]
   [2, 4, 6]
   [3, 4, 6]
   [1, 5, 6]
   [2, 5, 6]
   [3, 5, 6]
   [4, 5, 6]
```

Fig. 1. A short program in Mathematica™
showing a progression of advancing chords

It's just climbing up through the possibilities of what a mathematician would call six-choose-three[1], and here is a nice way of looking at it graphically. I'll put it in notes here, and I'll draw it on a six-note scale that can be found in the Slonimsky *Thesaurus* [D, F♯, G, B, C, E][2], connecting one chord to the

1. See *"binomial coefficient" or "n choose k"* in the Math Addendum.

2. Diatessaron Progression: Equal Division of Five Octaves in Twelve Parts in page 109 of [Slonimsky 1947].

To end this chapter, let's look at a progression that advances downwards, and at the same time we can go back to that list I made at the beginning of the chapter about height. I'll start with the eight chords that have sums of 10, then the seven that have sums of 9, the five that have sums of 8, and so on down to the chords with sums of 4 and 3, and why not, we can drop voices and end with the sums of 2, 1, and 0, which should make a real cadence. *Al niente* it's called sometimes.

```
 #1 [0,1,2] sum=3;   #2 [0,1,3} sum=4;   #3 [0,2,3] sum=5;
 #4 [0,1,4] sum=5;   #5 [1,2,3] sum=6;   #6 [0,2,4] sum=6;
 #7 [0,1,5] sum=6;   #8 [1,2,4] sum=7;   #9 [0,3,4] sum=7;
#10 [0,2,5] sum=7;  #11 [0,1,6] sum=7;  #12 [1,3,4] sum=8;
#13 [1,2,5] sum=8;  #14 [0,3,5] sum=8;  #15 [0,2,6] sum=8;
#16 [0,1,7] sum=8;  #17 {2,3,4] sum=9;  #18 [1,3,5] sum=9;
#19 [0,4,5] sum=9;  #20 [1,2,6] sum=9;  #21 [0,3,6] sum=9;
#22 [0,2,7] sum=9;  #23 [0,1,8] sum=9;  #24 [2,3,5] sum=10;
#25 [1,4,5] sum=10; #26 [1,3,6] sum=10; #27 [0,4,6] sum=10;
#28 [1,2,7] sum=10; #29 [0,3,7] sum=10; #30 [0,2,8] sum=10;
#31 [0,1,9] sum=10;
```

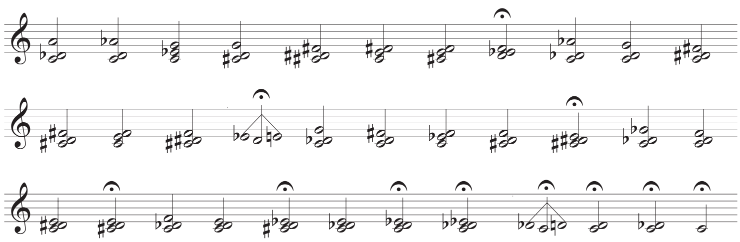


Fig. 6. Progression advancing downwards

(4 3 2 1)[1]. Since 1 + 2 + 3 + 4 = 10, the distance between the lowest and the highest notes is always a minor seventh and the outside voices never move. With a little effort I found a way to do it where each chord has three notes in common with the following chord.

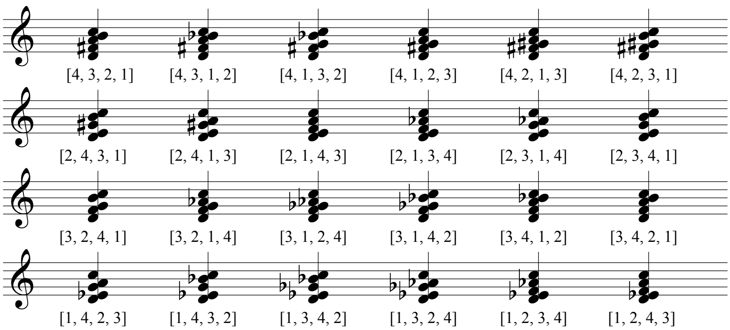


Fig. 1. Chords with the same intervals between adjacent notes

Here's the same set of permutations in numbers, as computed with Mathematica™. Notice that, computed in this way, sometimes all three of the inner notes change, and sometimes only one or two of them change. The previous example is probably more satisfying to a musician who thinks about voice leading, but there is a logic here too:

```
P := Permutations[1, 2, 3, 4]
Do[Print[0, Part[p[[i]], 1],
          Part[p[[i]], 1] + Part[p[[i]], 2],
          Part[p[[i]], 1] + Part[p[[i]], 2] +
                                          Part[p[[i]], 3],
          Part[p[[i]], 1] + Part[p[[i]], 2] +
                         Part[p[[i]], 3] + Part[p[[i]], 4]],
          i, 1, 24]
```

1. The Math Addendum will try to explain permutations.

systematic but most effective example is *Souvenir* by László Sáry[2], a piano piece that can have the effect of a lullaby.

Here is another advancing progression of five-note chords, where the intervals between adjacent notes must be either major thirds or perfect fourths and where the result is another way of advancing, higher and higher, little by little. Note that since we are just choosing combinations of two things, major thirds and perfect fourths, the five staves show one chord, then four chords, then six chords, then four chords, then one chord, which is a line from Pascal's triangle[3], the same numbers you get when you calculate the possibilities of heads and tails by throwing a coin four times. Of course, one could choose another pair of intervals and have a completely different sound, and one could work with chords of six notes or seven notes and have a much longer progression, but this much already demonstrate the principle.

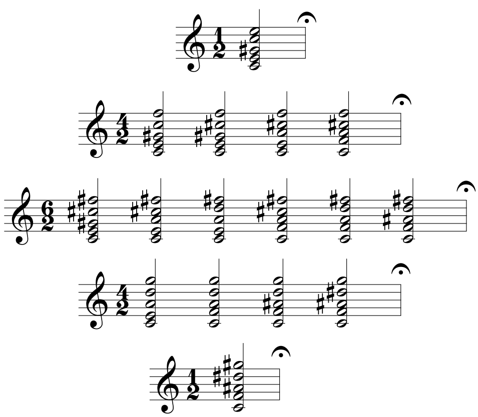


Fig. 4. Five-note chords with major thirds and fourths according to Pascal's triangle

2. László Sáry (1940–) is a Hungarian composer and pianist. As a pedagogue, he has developed the "Creative Music Activities".

3. See Math Addendum.

CONTINUATION

* How many four-note chords can be constructed with C as the lowest note and F as the highest? What are the adjacent intervals of these chords?

* *Pascal's Triangle* in the *Music for 88* used all the chords having only major seconds and minor thirds as adjacent intervals. How many five-note chords were there, and what did they sound like?

** In the chord (0, 1, 3, 6, 10) the intervals between adjacent notes are 1, 2, 3, and 4. How many other five-note chords with 0 as the lowest note and 10 as the highest note have these four adjacent intervals?

* My *360 Chords* for orchestra used all the seven-note chords where the intervals between adjacent notes were 3, 4, 5, 7, 8, and 9 semitones. With six adjacent intervals I could have formed 6! = 720 possible chords rather than 360, but I eliminated the ones with the small intervals in the bass, because the sound was too muddy. See the first bar of *360 Chords* in the illustration page of this chapter.

What was the range of the full chord?

** Construct the six four-note chords where the intervals between adjacent notes are 1, 2, and 3 semitones, with the bass note always 0 and the top note always 6. Is it possible to place these six chords in such an order that the order of the adjacent intervals will be completely different with each new chord? You will probably have to draw a graph to be sure.

** Consider a set of 5-note chords where the intervals between adjacent notes are {2, 3, 4|5, 7} i.e. 2, 3, 4 *or* 5, and 7. How many will have a height of 16 and how many will have a range of 17?

** Consider a set of 5-note chords where the intervals between adjacent notes are {2, 3, 4, 5 or 6}. Half will have a range of 14, and half will have a range of 15. How might they go together in such a way that they will sound like a complete family?

*** The family of five-note chords with {4, 5, 7, 8} as intervals between adjacent notes has 24 members. How many will be left if we eliminate all those containing an octave?

*** Construct chords of 9 notes, where the multiset of adjacent intervals is {2:3, 3:3, 4:2}. This notation indicates that the elements 2 and 3 have a multiplicity of 3 and should be used 3 times, while the element 4 has a multiplicity of 2. How many are possible with this constraint?

sums, all equivalent to 2 modulo 3. Below that you see the same configurations in musical notation (Fig. 7).

```
0 + 3 + 7 = 10        0 + 4 + 7 = 11
0 + 3 + 19 = 22       0 + 16 + 19 = 35
0 + 4 + 9 = 13        0 + 3 + 8 = 11
0 + 9 + 16 = 25       0 + 9 + 17 = 26
```

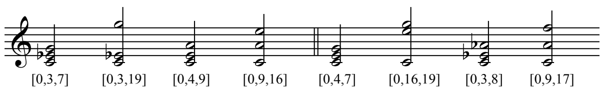


Fig. 7. Minor chords (1 mod 3) and major chords (2 mod 3)

Viewed in this way, major and minor triads are not in the same category, even though they belong to the same pitch class set, and one may ask what kinds of chords are in the third category, those whose sums equivalent to 0 modulo 3. Some of those chords can be found in the sequence that we already saw in Fig. 2, where we made a cycle following chords from all three categories (Fig. 8).

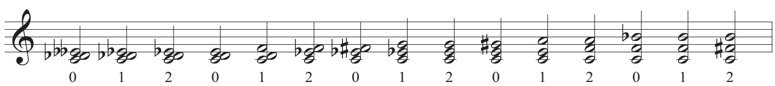


Fig. 8. Chords equivalent to 0, 1, and 2 modulo 3

It made sense with three-note chords to consider their sums modulo 3, because any three-note chord in any transposition will retain its sum modulo three. Similarly with four-note chords, it is logical to consider their sums modulo 4. A chord with a sum of 11, when transposed, may have a sum of 15 or 27, but the sum will always be equivalent to 3 modulo 4. I have spent a lot of time and used up a lot of music paper trying to study the chords equivalent to 0, 1, 2, and 3 modulo 4 over the years, and I have been particularly interested in the chords with sums equivalent to 1 modulo 4, because and that makes the voice leading very awkward, so I decided to take another approach. I asked the computer to compute all the four-note chords with sums of 20 that had no semitones as adjacent intervals and that could fit into a single octave. I then made graphs to figure out how to put them together with minimal changes, that is, with one voice moving up a semitone and another voice moving down a semitone. Then I did the same thing with chords having sums of 21, 22, and 23, so that I would have a collection of the four different families, all made with the same procedure, and this is what I found (Fig. 10).

Fig. 10. Families of four-note chords according to their sums modulo 4, connected by minimal differences

It's probably better if you just listen to those results without paying any attention to my own reactions, but when you're ready, I'll give you my observations in the next paragraphs.

Well, the most obvious thing about the first music, the 0 modulo 4 chords, is the chords where all four notes are on the same whole tone scale. There are four of those, but I see several minor sevenths too, Forte 4-26 (0,3,5,8), and almost all the chords are palindromes, meaning that the inversions are the same as the originals. I don't think that can happen when the sum is equivalent to 1 or 3 mod 4, though I wouldn't know how to prove it[3].

The second phrase, the chords with sums equivalent to 1 mod 4, is most characterized by the dominant sevenths, which account for four of the 11 chords. Several others are major triads with an added second degree, and sometimes, as with the second and third chords, they are even in the same key. That C dominant seventh, the sixth chord, for example, could resolve very nicely to the first chord. Whole tone chords can not appear, since the sum of four odd numbers or four even numbers will always be 0 or 2 mod 4, and I don't see any palindromes either.

The third phrase, the chords with sums equivalent to 2 mod 4, brings more whole tone chords, but also a diminished seventh, and there at the end, we have another palindromic chord, commonly known as a major seventh, which makes a traditional cadence. The very first chord is a palindrome too, and I think those palindromes are balanced in a way that gives this music, like the 0 mod 4 music, a special equilibrium. It feels kind of settled, whereas the chords in the 1 mod 4 music seem somehow off center.

Finally, with the fourth phrase, the chords with sums equivalent to 3 mod 4, things seem to be moving again. For me those four Tristan chords, chords 3, 6, 8, and 9, set the atmosphere, and all the other chords are ones you don't hear very much. If Wagner were still alive, he might well want to write his music by calculating chords with sums equivalent to

3. Actually I did. See the proof in the Math Addendum.

ning of homometry in general. Any two chords are homometric if they have exactly the same interval content, and this is quite possible with pairs of chords with more than four notes. Forte found three different pairs of five-note chords with the same intervallic content, but rather than giving you all those numbers, I have simply drawn these five-point formations on 12-point circles (Fig. 12). I'll add the numbers that mark the distances, which are rather irregular. There are 10 distances between the pairs of notes, and some intervals are more present than others.

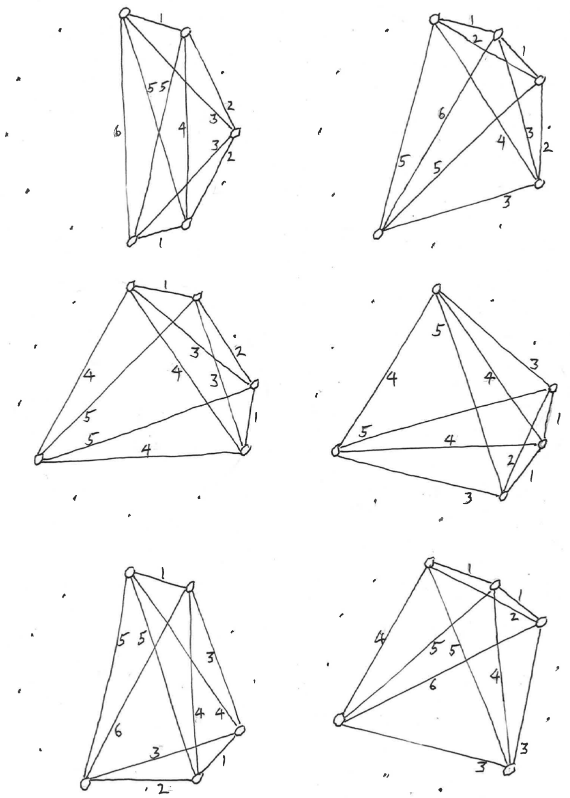


Fig. 12. Forte 5-Z12, 5-Z36; Forte 5-Z17, 5-Z37; Forte 5-Z18, 5-Z38

You may think that these chords sound so homogeneous just because I kept C and E as the lower notes in all cases, but here are some other five-note chords that come from Forte 5-Z18 and Forte 5-Z38 and they sound pretty similar too (Fig. 18).

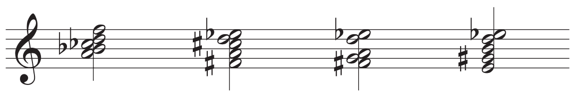


Fig. 18. Other chords from the same pair

They are kind of squeezed though. They will sound better if the voices are spread out a bit so that, as Messiaen would have said, the "divine light" can shine through (Fig. 19).

Fig. 19. Same chords spread out

Well, I'm not sure that the "divine light" is shining through, but the music certainly sounds better, and still quite homogeneous, and don't forget that this is just the tip of the iceberg. Even the world of the all-interval tetrachord remains little explored in music, and homometric formations of more than four-notes are still virgin territory for composers.

the blocks together when they have minimal differences and end up with long chains, or sometimes one single chain, but in this solution very few links of that sort are possible. The block (1,4,7,9) has three notes in common with the block (4,5,7,9), and 11 other pairs of blocks have three notes in common, but no chains are possible. After confirming this, I also asked the computer if there were cases where three blocks contained only six notes and what did I find? Eureka! Ten such formations! And in each case the three blocks all had two notes in common with one another. I won't write out all ten, but here are a few, which I wrote out on a new scale about two octaves in range, in order to spread the intervals out a bit. I've written this music in quarter notes with repeat signs, because I often find that when chords are very similar it is nice to let them repeat at a faster tempo (Fig. 7). I also drew the structure in triangles to see better how the same notes interconnect between the chords (Fig. 8).

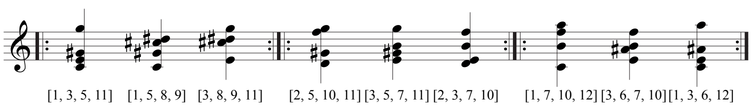


Fig. 7. Sets of three chords containing six notes

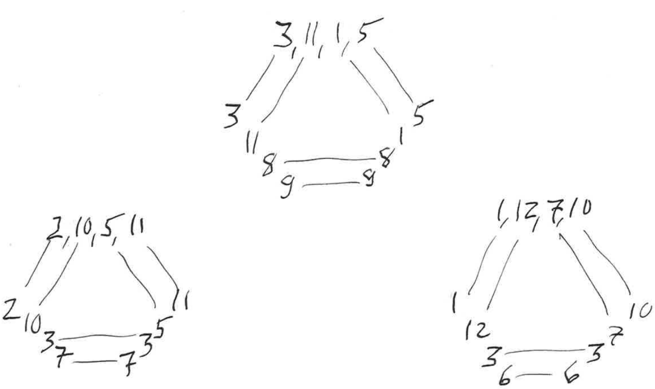


Fig. 8. Sets of three chords formed with six notes

CONTINUATION

* Divide the notes/numbers 1, 2, 3, 4, 5, 6, 7, 8 into subsets of four notes forming parallel classes.

** How many ways are there to do this?

** The fourth and last solution for an (8,4,3) design contains these 14 blocks. Only two pairs come together to form a parallel class. Try to find them.

{{0,1,2,3}, {0,1,4,5},{0,1,6,7}, {0,2,4,6}, {0,2,5,7}, {0,3,4,7}, {0,3,5,6}, {1,2,4,7}, {1,2,5,6}, {1,3,4,6}, {1,3,5,7}, {2,3,4,5}, {2,3,6,7}, {4,5,6,7}}

*** Make a graph of these 14 blocks when they have two or more notes in common. Note that by definition every pair occurs in three different blocks.

*** Here are the 33 blocks of the Morales and Velarde (12,4,3) solution. Below is my program that will divide the 33 blocks into 11 parallel classes, but you might want to write your own program.

```
{{{0, 1, 2, 3},  {0, 1, 4, 5},   {0, 1, 6, 7}},
 {{0, 2, 4, 10}, {0, 2, 5, 11},  {0, 3, 6, 8}},
 {{0, 3, 7, 9},  {1, 2, 7, 8},   {1, 2, 6, 9}},
 {{1, 3, 5, 10}, {1, 3, 4, 11},  {4, 5, 6, 7}},
 {{2, 3, 8, 9},  {2, 3, 10, 11}, {1, 6, 8, 11}},
 {{1, 7, 9, 10}, {1, 4, 9, 10},  {1, 5, 8, 11}},
 {{0, 4, 9, 11}, {0, 5, 8, 10},  {0, 6, 9, 11}},
 {{0, 7, 8, 10}, {8, 9, 10, 11}, {6, 7, 10, 11}},
 {{4, 5, 8, 9},  {3, 5, 7, 9},   {3, 4, 6, 8}},
 {{2, 5, 7, 11}, {2, 4, 6, 10},  {3, 5, 6, 10}},
 {{3, 4, 7, 11}, {2, 4, 7, 8},   {2, 5, 6, 9}}}
```

"0 to 11"

12-tone theory has used the group Z/12, identifying the notes with the elements:

0 = C, 1 = C♯, …, etc. up to 11 = B

But a number can also mean an interval, e.g., the number 5 is a perfect fifth. The sums of notes or intervals are reduced modulo 12, which means that they equal the remainder when divided by 12.
4 + 10 = 2 mod 12 because 14 divided by 12 leaves a remainder of 2. Musically this represents adding 10 semitones (a minor seventh) to E (4) to get D (2).
In a group the inverse of an element gives zero when added to it: 4 + 8 = 0. Very conveniently, this coincides with the inverse of an interval. So a minor sixth (8) is the inverse of a major third (4).
A common representation of the elements/notes is a chromatic circle with 12 points.

"four-note chords that are palindromes are 0 or 2 modulo 4."

A four-note chord is a palindrome if it is equal to its inversion. [0,1,4,5] is an example. And if some chord [a,b,c,d] is palindromic then b – a = d – c
Let's study now the expression b – a.
If b – a is even then b – a = 2k. Adding 2·a to both sides we get a + b = 2(k+a), therefore a + b is even too.
If b – a is odd then b – a = 2k+1. Adding 2·a to both sides we get a + b = 2(k+a) +1, therefore a + b is odd.
We have proved that a + b and c + d are both even or both odd, and so the total a + b + c + d must be even. The sum of the chord is thus 0 or 2 mod(4).

“prime with” or “coprime”

Two integers are said to be coprime if their only common divisor is 1. Twelve and five are coprime. Twelve and ten are not coprime because 2 is a common divisor of both.
If a and b are coprime, it can also said that “a is prime with b”.

“permutations”

When we talk about reordering the elements of a set, we are talking about permutations. If a set has n elements, the number of permutations is

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1.$$

This is could be illustrated with a five elements set {a, b, c, d, e}. For the first element of the permutation we can choose from a, b, c, d, or e, so we have five options. After that we have four elements left to choose from (if you chose ‘a’ you can now choose b, c, d, or e). So far we have 5 × 4 partial permutations. With 3 elements left, we have three choices, with 2 elements remaining, two choices, and we end up with the one element left.
The whole process gives us 5 × 4 × 3 × 2 × 1 = 5! = 120 permutations.
When the elements to choose from are repeated, the total number of permutations is reduced, because one cannot distinguish one repeated element from another. If there are, for example, 3 a’s, we have to divide the total number of permutations by 3! = 6.

“binomial coefficient” or “n choose k”

When we choose k elements from a set of n, regardless of the order, we form a combination, which is an unordered permutation. To choose the first element of a subset with kelements we have n options, after which we have (n-1) options, then (n-2) and so forth, down to (k+1).
That gives us $n \times (n-1) \times \ldots \times (k+1)$ permutations of k elements. But we are not interested in the order of the elements, so let’s divide by the number of permutations of k elements, k!

At the end, the so called *binomial coefficient* (n-choose-k) is $[n\times(n-1)\times\ldots\times(k+1)]/k!$

When we represent these coefficients in rows, starting with n=0 and giving k all the values possible up to n, we obtain Pascal’s triangle, after the French mathematician Blaise Pascal, but also studied by others before him in India, China, etc:

```
n=0, k=0                              1
n=1, k=0,1                          1   1
n=2, k=0,1,2                      1   2   1
n=3, etc                        1   3   3   1
n=4                           1   4   6   4   1
n=5                         1   5  10  10   5   1
n=6                       1   6  15  20  15   6   1
```

The “20” in the seventh row is the answer to 6-choose-3.

A note on notation

This book uses numbers to represent different things: factors, pitch classes, actual notes, intervals, elements of a set, etc. Thus, depending on the context, some numbers inside parenthesis may mean a pitch class set or a block-design. We distinguish between normal parentheses, curly brackets and square brackets with the hope that the meaning is more clear. Below we summarize the principal examples of notation used throughout the book (the '≡ means "is identical to"):

{2:3, 5, 7} ≡ {2, 2, 2, 5, 7} ≡ Multiset, i.e. a set where each element has a multiplicity.

(0,3,7) ≡ the prime form of a pitch class set (pc set), following Allen Forte's procedures. (0,3,8) and (0,4,7) are in the same pitch class set, but not in prime form. Notice that there are no spaces after the commas.

[4, 7, 12] ≡ a sequence of notes forming a chord with actual pitches, measured from middle C = 0. This notation is used for calculating heights with no octave reduction allowed, i.e. [4, 7, 12] is not equivalent to [0, 4, 7].

(2 1 4 3) ≡ a permutation of an ordered set (1 2 ... 4). No commas.

(6,3,2) ≡ a block design of 6 elements, divided into subgroups of 3, with every pair of elements comes together 2 times. Again no spaces after the commas.